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Signal and Noise Models Arising from the Theory of Errors

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Section 1: Research by Dr. LePage under the support of grant ONR N00014-85-K-0150.

A. **Series constructions of stable random processes and invariance principles:** Dr. LePage [23, appendix] has developed series constructions of stable processes (and more generally operator stable processes) now generally known as the LePage Representation. This construction is a widely used tool in the study of stable processes and has been used to obtain new results ranging from path properties (M. Marcus, G. Pisier), extrema of stable processes (G. Samorodnitsky), Bootstrap (K. Knight), stable measures of balls (M. Lewandowski, et. al.) and a host of other applications (Ledoux, Tallagrand). Recently Dr. LePage [6] generalized and simplified the series construction of stable processes by proving that it follows from the method of marking of a Poisson process.

Another fundamental use of series constructions is made by Dr. John Kinateder in his Ph.D. thesis An Invariance Principle Applicable to the Bootstrap [9], written under the direction of Dr. LePage and supported by the grant. In the thesis Dr. Kinateder (now employed at Battelle) establishes a series construction of the LePage-type for sums of i.i.d. random variables in the domain of attraction of a SαS Symmetric alpha stable) distribution and proves a new type of invariance principle describing the convergence of such constructions to the LePage series for the limit laws. This invariance principle is applicable to describing the joint behavior of the sample sums, re-sampled sums, and order statistics, and will provide the basis for entirely new conditional inference approach to statistical analysis in stable noise environments (see Seamless Resampling below).

B. **Gaussian Slicing:** Dr. LePage [4, appendix] proved that every

SαS random vector or process, irrespective of time domain or range, has the distribution of a mixture of Gaussian distributions. That is, the probability space may be sliced into (infinitely many) slices on each of which the stable process is conditionally a Gaussian process (dependent in a particular way upon that slice). Dr. LePage [6] has recently extended this to the underlying stable Poisson point process (see Point Process Approach to Stable Processes below). Using Gaussian slicing, powerful methods available from Gaussian theory may be brought to bear on SαS processes with sometimes surprising results. Dr. LePage [3] [4] exploited Gaussian slicing to establish $E(X^2(t) \mid X(t-d), X(t-2d))$ is finite a.s. for the Fourier transforms $X(t) = \int \exp(itx)\, Z(dx)$ of complex (conjugate) symmetric independent increments stable noise, even though $EX^2(t)$ is infinite for every $x < 2$. This result, the first of its type, means that second order prediction is valid for such processes and has precipitated a number of papers (Cambanis, Taqqu). Other researchers have applied slicing to Slepian type inequalities [G. Samorodnitsky], and most recently to the stable measures of balls (M. Lewandowski, et. al.). The latter actually obtains the Gaussian result also by these methods. Aside from such technical uses of slicing it has deep implications for prediction and estimation in SαS environments. In an invited talk before the Conference on Stable Measures and Extremes held in conjunction with the regional IMS Boston Meeting (1987), Dr. LePage presented results which show that prediction for the SαS case is, by Gaussian slicing, a Bayes problem with the data consisting of a single realization in which one observes the path of a Gaussian process whose covariance operator is itself a random α/2 stable covariance process. It follows that prediction can be viewed in two steps: (a) use the data (and the α/2 stable model!) to estimate a covariance operator (i.e. to estimate the slice) and then (b) use the best Gaussian predictor for that estimated slice.

Actual simulations done on a computer show that for $\alpha = 1$ (Cauchy processes) the resulting estimator of location for i.i.d. data is virtually identical to the median! So the approach does appear to be correct. New results of Dr. LePage [14] show how a Bootstrap-like re-sampling method can automatically calibrate the estimation procedure for the index of stability and bypass all the specialized calculations of the Bayes approach. These results appear in the Proceedings of the IMS Bootstrap Conference recently held at MSU on May 15-16, 1990 (see Seamless Resampling below).

C. **Point process approach to stable processes:** Dr. LePage [6], recently proved that Gaussian slicing holds also for the stable Poisson point processes which underly stable processes. That is, $\alpha < 2$ symmetric stable (with respect to set-union of points and scale change) Poisson point processes may be represented as mixtures of Gaussian point processes. Furthermore, as also proved in [6], stable Poisson point processes exist for all indices of stability $\alpha = 0$ and have applications to e.g. density estimation and super-efficient estimation of location. Thus stable noise processes are identified with the summation operator applied to stable point processes, but if we look beyond sums to other functions of random point disturbances then all indices of stability $\alpha = 0$ become relevant and have varying applications. This perspective will probably come to dominate the study of stable processes.

D. **Seamless Resampling.** Dr. LePage [7], and LePage and Podgorski [14] develop a bootstrap-like resampling method for the sample average of i.i.d. r.v. in the domain of attraction of a symmetric stable law of arbitrary index $\alpha < 2$ to which they give the name Seamless Resampling. It recovers the limiting

distribution of contrasts in regression, conditional on the vector of absolute values of these same errors. This is an improvement on what was earlier presumed to be the goal: recovery of the unconditional limit law and leads to narrower confidence intervals. The conditional inference possible by seamless resampling is not ordinarily considered in the $\alpha = 2$ case and is an entirely new aspect of inference for SαS discovered under ONR N00014-85-K-0150. With the seamless resampling procedure it is possible to use resampling to automatically identify the limit model and adjust for the correct values of the unknown nuisance parameters without explicitly incorporating any such information in the description of the statistical method itself. Contrast these results with what was previously known. Athreya proved that ordinary Bootstrap (not seamless resampling!) fails for the case of the sample average in this case if $\alpha < 2$. Gine and Zinn proved that in a certain sense Bootstrap requires finiteness of the second moment. All of this makes the seamless resampling result extremely interesting since one of the difficulties of developing statistical methods capable of dealing with stable noise has been to work out the needed distribution theory. With resampling methods it may be possible to bypass this step altogether.

E. **Optimality methods based on expected logarithm.** Entropy methods such as the Burg algorithm used successfully in spectral density estimation often have many useful applications but little theory behind them. To a large degree it is not known why they work. Dr. LePage has been studying the connections between entropy maximization or minimization and the growth of random products. This has connections with the work of L. Breiman (4th Berkeley) on the Kelly Principle and T. Cover on portfolio optimization. In such problems the idea is to control a random process so that it grows toward infinity most rapidly, or decays

toward zero least rapidly. Dr. LePage and Dr. Schreiber [5] connected this idea with sequential estimation. They used martingale decomposition methods to establish rapid growth of a partial product of random variables indexed by parameters when these parameters are at each stage chosen to maximize the conditional expectation of the logarithm of the next term of the product given the information then available. They applied this result to sequentially estimating a parameter in such a way as to make the product of the likelihoods decrease least rapidly. Dr. LePage [12] addressed a similar problem for particular diffusions using the Ito calculus to develop stronger forms of the rapid growth result. The basic conclusion is that under appropriate conditions the drift term in the Ito differential of the log of the process should be maximized in order that the process grow most rapidly to infinity. Dr. LePage [11] considered random products plotted at random times and proved that under some conditions the product tends most rapidly to infinity in the random time scale if at each stage one maximizes the ratio of the conditional expected log of the next term of the product to the conditional expected time to obtain it. This result is being studies for its potential application to real-time estimation and control.

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Section 2: Research done by Dr. Mandrekar under the support of grant ONR N00014-85-K-0150.

The research which was proposed under the grant has several components; Linear prediction of SαS processes, Multiparameter processes, robust detection, and higher order spectra. In the linear prediction it was demonstrated by Dr. Mandrekar [4] that the time domain and spectral domain theories have to be developed separately under the then existing techniques. It was shown [4] that all existing work on prediction problem for strongly harmonizable SαS processes can be derived by adapting classical methods. However to handle general problem one needs Generalized spectrum. This was introduced in [4].

As isometries in Lp have no orthogonality properties except when $p = 2$, the definition of spectrum covered non-stationary processes thus giving a new analytic tool even in the case $p = 2$. In general, it is also shown [9] that SαS processes are images of L^2 processes. The existing orthogonality techniques used in the field were shown to be a special case of semi inner product of Lumer and the scope of these techniques in applications was expanded in ([12] [15]).

The time domain analysis was connected to the geometry of linear subspaces generated by the process and this was used to study ARMA processes in exchangeable random variables ([8]). Dr. Mandrekar has been successful in computing the Generalized Spectrum for moving averages and thus the techniques for the development of relation between the time domain and the spectral domain for SαS processes are ready. To handle some nonlinear problems the structure of processes of special type (n-ple Markov) have been established [7]. these relate to the spherical averages of isotropic multidimensional SαS processes similar to Levy Brownian motion. Thus one can study prediction of spherical

averages of multiparameter SαS processes. In addition, the basic analytic tools on Beurling Theorem, inner outer factorization and the structure of invariant subspace of L^2 over a polydiscs have been established ([1], [5]) by Dr. Mandrekar. In addition, Drs. Kallianpur and Mandrekar established the time domain analysis of multiparameter processes. Techniques for the study of limit theorems for non-linear functional of i.i.d. triangular arrays have been established starting with [3]. In the stable case, these give operator stable processes with diagonal operators i.e. multi-dimensional stable with components stable with different index. These techniques will enable one to study limit theorems for higher order spectra and those related to ARMA processes with noise in the domain of attraction of stable law.

The study in ([11], [14]) is related to robust detection of signals with finite fisher information. In particular, this includes SαS product measures. The work in statistical analysis of Markov space-time processes have been initiated by Dr. Mandrekar [10] which is motivated from his study related to Markov type processes [2]. For this initial study was carried out for point process techniques in statistical analysis [6]. However, the general problem of space time processes seems to be open even for Markov equilibrium measures. This was studied by Dr. Zhang [13] under the direction of Dr. Mandrekar.

Reports supported under the grant ONR N00014-85-K-0150.

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